Journal of Statistical Physics, Vol. 40, Nos. 1/2, 1985

A Note on the Boltzmann Equation for Hard Spheres

Y. Pomeau[1]

Received December 20, 1984

A simple form of the Boltzmann kinetic equation for hard spheres is proposed.

In the course of investigations on the structure of shock waves at large Mach number, I have been led to look for a form of the Boltzmann equation for hard spheres that were as simple as possible. The form that I have found is, as far as I know, original and could be useful (for instance) in numerical researches on the kinetic theory of the hard sphere gas. Let $f(\mathbf{c}, t)$ be the velocity distribution, the Boltzmann kinetic equation for the hard sphere system reads[1]

$$\frac{\partial f}{\partial t} = B[f, f] \tag{1}$$

where

$$B = B_l + B_g$$

with

$$B_l \equiv -4\pi \int d\mathbf{c}_1 \, f(\mathbf{c}_1) \, |\mathbf{c} - \mathbf{c}_1| \, f(\mathbf{c}) \tag{2}$$

and

$$B_g \equiv \int d\mathbf{c}_1 \int d\hat{n} \, f\left(\frac{\mathbf{c} + \mathbf{c}_1}{2} + \frac{\hat{n}}{2} |\mathbf{c} - \mathbf{c}_1|\right) \times f\left(\frac{\mathbf{c} + \mathbf{c}_1}{2} - \frac{\hat{n}}{2} |\mathbf{c} - \mathbf{c}_1|\right) |\mathbf{c} - \mathbf{c}_1| \tag{3}$$

[1] Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France.

0022-4715/85/0700-0361$04.50/0

$\hat{n}$ being the unit vector $[\int d\hat{n} = 4\pi]$. A few simple transformation give the following expression for the loss term (B_l), in the case of isotropic distributions (f depends on $|\mathbf{c}|$ only):

$$B_l = -8\pi^2 \int_0^\infty dc_1\, c_1^2 f(c_1) f(c) L(c, c_1)$$

where

$$L(c, c_1) = \frac{2}{3c}(3c^2 + c_1^2) \qquad \text{if} \quad c \geqslant c_1$$

and

$$\frac{2}{3c_1}(3c_1^2 + c^2) \qquad \text{if} \quad c \leqslant c_1$$

The gain term B_g is less easy to transform. We shall obtain two successive expressions, the first one valid for an arbitrary distribution [Eq. (4) below], the second one for an isotropic distribution [Eqs. (5) and (8) below]. This last one only is interesting for computational purposes, because it reduces (3) to a two-dimensional integral, although in general the form given in (4) does not lower the number of integration variables.

Let us write $\int d\hat{n}(\cdots)$ as $\frac{1}{2}\int d\mathbf{n}\, \delta(n^2 - 1)(\cdots)$ δ = Dirac function, and make in (3) the changes $\mathbf{c}_1 \to \mathbf{c} + \boldsymbol{\xi}$ and $\mathbf{n} \to \mathbf{n} = \mathbf{N}/\xi$. This yields from (3)

$$B_g = \frac{1}{2}\int d\boldsymbol{\xi} \int d\mathbf{N}\, \delta(N^2 - \xi^2) f\left(\mathbf{c} + \frac{\boldsymbol{\xi} + \mathbf{N}}{2}\right) f\left(\mathbf{c} + \frac{\boldsymbol{\xi} - \mathbf{N}}{2}\right)$$

Replace now $\boldsymbol{\xi}$ by $\boldsymbol{\xi}^* = \boldsymbol{\xi} + 2\mathbf{c}$ as integration variable, and then take $\mathbf{x}_\pm = (\boldsymbol{\xi}^* \pm \mathbf{N})/2$ as integration variables instead of $\mathbf{N}$ and $\boldsymbol{\xi}^*$

$$B_g = 4\int d\mathbf{x}_+ \int d\mathbf{x}_-\, \delta(\mathbf{c}\cdot(\mathbf{x}_+ + \mathbf{x}_-) - c^2 - \mathbf{x}_+ \cdot \mathbf{x}_-) f(\mathbf{x}_+) f(\mathbf{x}_-) \tag{4}$$

This is our general expression for B_g. Let us now restrict ourselves to the case of isotropic distributions. In that case, B_g depends on $|\mathbf{c}| = c$ only, and one has

$$B_g = \frac{1}{4\pi}\int d\hat{c}\, B_g(c)$$

where $\hat{c}$ is the unit vector such that $\mathbf{c} = \hat{c}c$. Thus (4) can now be written as

$$B_g = 4\int_0^\infty dx_+\, x_+^2 f(x_+) \int_0^\infty dx_-\, x_-^2 f(x_-) M(x_+, x_-; c) \tag{5.a}$$

where

$$M(x_+, x_-; c) = \frac{1}{4\pi} \int d\hat{c} \int d\hat{x}_+ \int d\hat{x}_- \, \delta(\mathbf{c} \cdot (\mathbf{x}_+ + \mathbf{x}_-) - c^2 - \mathbf{x}_+ \cdot \mathbf{x}_-) \quad (5.b)$$

$\hat{x}_\pm$ being the unit vector(s) parallel to $\mathbf{x}_\pm$. Performing in (5.b) the $\hat{c}$ integration, one finds

$$\frac{1}{4\pi} \int d\hat{c} \, \delta(\mathbf{c} \cdot (\mathbf{x}_+ + \mathbf{x}_-) - c^2 - \mathbf{x}_+ \cdot \mathbf{x}_-)$$
$$= \frac{1}{2c\,|\mathbf{x}_+ + \mathbf{x}_-|} \chi_{[-1,+1]} \left(\frac{c^2 + \mathbf{x}_+ \cdot \mathbf{x}_-}{c\,|\mathbf{x}_+ + \mathbf{x}_-|} \right) \quad (6)$$

where $\chi_{[-1,+1]}(u)$ is the characteristic function of the $[-1, +1]$ interval. Its value is $+1$ if $-1 \leqslant u \leqslant +1$ and zero otherwise. From (5) and (6)

$$M(x_+, x_-; c) = \frac{4\pi^2}{c} \int_{-1}^{+1} \frac{dv}{(x_+^2 + x_-^2 + 2x_+ x_- v)^{1/2}}$$
$$\times \chi_{[-1,+1]} \left[\frac{c^2 + x_+ x_- v}{c(x_+^2 + x_-^2 + 2x_+ x_- v)^{1/2}} \right] \quad (7)$$

This last integral can be computed, and one obtains our final result. The support of $M(x_+, x_-; c)$ as a function of x_+ and x_- is the quarter plane $x_\pm \geqslant 0$. From the energy conservation $x_+^2 + x_-^2 \geqslant c^2$, so that M is nonzero outside the quarter circle $x_+^2 + x_-^2 = c^2$ only. The consideration of the argument of the characteristic function in (7) leads one to introduce the quartic Q of Cartesian equation $x_+^2 x_-^2 + c^4 = (x_+^2 + x_-^2)\, c^2$, and to divide the quarter plane into three regions: $\Gamma_{1,a-b}$ are in between Q and the circle of Cartesian equation $x_+^2 + x_-^2 = c^2$. In $\Gamma_{1,a} x_+ > x_-$ although $x_+ \leqslant x_-$ in $\Gamma_{1,b}$. Let furthermore Γ_2 be the region outside of Q.
Thus

$$\text{in } \Gamma_{1,a} \qquad M(x_+, x_-; c) = \frac{2\pi^2}{x_+ c} \quad (8.a)$$

$$\text{in } \Gamma_{1,b} \qquad M(x_+, x_-; c) = \frac{2\pi^2}{x_- c} \quad (8.b)$$

and

$$\text{in } \Gamma_2 \qquad M(x_+, x_-; c) = \frac{\pi^2}{x_+ x_- c}$$
$$\times \left[(x_+^2 + x_-^2 + 2w^2)^{1/2} - (x_+^2 + x_-^2 - 2w^2)^{1/2} \right] \quad (8.c)$$

where $w^2 = [c^2(x_+^2 + x_-^2) - c^4]^{1/2}$. As a function of $x_\pm$, $M(x_+, x_-; c)$ is continuous, but has discontinuous first derivatives in Q and on the first bisectrix.

REFERENCES

1. S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Chap. V (Cambridge University Press, Cambridge, 1972).